LECTURE NOTES ON SCHAUDER ESTIMATES

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1. Introduction

In the lecture notes we will prove the Schauder estimates for viscosity solutions. Throughout the notes we will always assume that $a_{ij} \in C(B_1)$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for any } x \in B_1 \text{ and any } \xi \in \mathbb{R}^n$$

for some positive constants λ and Λ and that f is a continuous function in B_1 .

The notes are organized as follows. An equivalent characterization of Hölder regularity is given in section 1. In section 2, we will give an approximation result which plays an important role in the disscussion of regularity theory. Section 3 deals with the Schauder estimates for viscosity solutions.

2. Hölder Regularity

In the following, we give one way to approach C^α,$C^{1,\alpha}$ and $C^{2,\alpha}$ estimates($\alpha \in (0,1)$) via approximation by constants, planes and paraboloids respectively.

Lemma 2.1. *A function u is in C^α if and only if for all $x \in \Omega$, there is some constant C_x and a uniform constant K such that*

$$\|u - C_x\|_{L^\infty(B_r(x))} \leq Kr^\alpha.$$

Furthermore, if $\sup_\Omega |C_x| + K \leq M$ for all $x \in \Omega$, then $\|u\|_{C^\alpha(\Omega)} \leq M$.

Proof : This is an immediate consequence of the definition of Hölder regularity. The picture is that u is trapped between two α-polynomials of opening K at every point.

Lemma 2.2. *A function u is in $C^{1,\alpha}(\Omega)$ if and only if for any $x \in \Omega$, there is some affine approximation $l_x(y) = a_x + \langle b_x, y - x\rangle$ and a uniform contant K such that*

$$\|u - l_x\|_{L^\infty(B_r(x))} \leq Kr^{1+\alpha}.$$

Furthermore, if $sup_\Omega|a_x| + sup_\Omega|b_x| + K \leq M$,then

$$\|u\|_{C^{1,\alpha}(\Omega)} \leq M.$$

The picture is that u is trapped between two $\alpha+1$-polynomials of opening K at every point.

Proof : First, if $u \in C^{1,\alpha}(\Omega)$, then $\forall x \in \Omega$, let $l_x(y) = u(x) + \langle Du(x), y - x\rangle$ and $K = [Du]_{C^\alpha(\Omega)}$. Then for any $y \in B_r(x)$, by Taloy's expansion we have

$$|u(x) - l_x y| = |\langle Du(\xi) - Du(x), y - x\rangle| \leq Kr^{1+\alpha}.$$

Next, if we let $f_r(x) = \frac{1}{r}f(rx)$ be the linear rescaling of any function f. Let x and y be two points a distance r apart, and by translation assume that x and y are symmetric about 0. By hypothesis,

$$\|l_{x,r} - l_{y,r}\|_{L^\infty(B_1)} \leq \|u_r - l_{x,r}\|_{L^\infty(B_1)} + \|u_r - l_{y,r}\|_{L^\infty(B_1)} \leq 2Kr^\alpha.$$

Observing that the L^∞-norm of any affine function on B_1 controls its coefficients, we see that $|Dl_x - Dl_y| \leq Kr^\alpha$. The last statement follows because $sup_\Omega|a_x| + sup_\Omega|b_x| + K$ is the $C^{1,\alpha}$-norm of u.

Remark 2.3. *In the proof of above lemma, we need to point out that if $l_x(y) = a_x + \langle b_x, y - x\rangle$ satisfies $\|u - l_x\|_{L^\infty(B_r(x))} \leq Kr^{1+\alpha}$, then we can immediately get u is differentiable in Ω and $Du(x) = Dl_x = b_x$. Additionally, if $\alpha = 0$, this implies $u \in C^{0,1}$, not C^1. Hence the proof dosen't work in this case.*

Lemma 2.4. *A function u is in $C^{2,\alpha}(\Omega)$ if and only if for any $x \in \Omega$, there is some quadratic polynomial $P_x(y) = a_x + \langle b_x, y - x\rangle + \frac{1}{2}(y - x)C_x(y - x)^T$ and a uniform contant K such that*

$$\|u - P_x\|_{L^\infty(B_r(x))} \leq Kr^{2+\alpha}.$$

Furthermore, if $sup_\Omega|a_x| + sup_\Omega|b_x| + sup_\Omega|C_x| + K \leq M$,then

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq M.$$

The picture is that u is trapped between two $\alpha + 2$-polynomials of opening K at every point.

Proof : The proof is similarily as the affine case. But in this case, we should choose $f_r(x) = \frac{1}{r^2}f(rx)$ as the quadratic rescaling of a function f. Moreover, if $\alpha = 0$, $\|u - P_x\|_{L^\infty(B_r(x))} \leq Kr^2$ can only implies $C^{1,1}$, not C^2. A detailed proof is left for you guys to replenish.

3. Approximation Result

Lemma 3.1. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f$$

in B_1 with $|u| \leq 1$ in B_1. Assume for some $0 < \varepsilon < \frac{1}{16}$,

$$\|a_{ij} - a_{ij}(0)\|_{L^n(B_{\frac{3}{4}})} \leq \varepsilon.$$

Then there exists a function $h \in C(\overline{B}_{\frac{3}{4}})$ *with* $a_{ij}(0)D_{ij}h = 0$ *in* $B_{\frac{3}{4}}$ *and* $|h| \le 1$ *in* $B_{\frac{3}{4}}$ *such that*

$$|u - h|_{L^\infty(B_{\frac{1}{2}})} \le C\{\varepsilon^\gamma + \|f\|_{L^n(B_1)}\}$$

with $C = C(n, \lambda, \Lambda)$ *is a positive constant and* $\gamma = \gamma(n, \lambda, \Lambda)$.

Proof : Solve for $h \in C(\overline{B}_{\frac{3}{4}}) \cap C^\infty(B_{\frac{3}{4}})$ such that

$$\begin{aligned} a_{ij}(0)D_{ij}h = 0 \qquad & in \quad B_{\frac{3}{4}}, \\ h = u \qquad & on \quad \partial B_{\frac{3}{4}}. \end{aligned}$$

The maximum principle implies that $|h| \le 1$ in $B_{\frac{3}{4}}$. Note that u belongs to $S(\lambda, \Lambda, f)$ in B_1. Corollary 5.11 in H-L implies that $u \in C^\alpha(\overline{B}_{\frac{3}{4}})$ for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$ with the estimate

$$\|u\|_{C^\alpha(\overline{B}_{\frac{3}{4}})} \le C(n, \lambda, \Lambda)\{1 + \|f\|_{L^n(B_1)}\}.$$

By Proposition 4.13 in Caffarelli or Lemma 1.35 in H-L, we have

$$\begin{aligned} \|h\|_{C^{\frac{\alpha}{2}}(\overline{B}_{\frac{3}{4}})} &\le C(n, \lambda, \Lambda)\|u\|_{C^\alpha(\partial B_{\frac{3}{4}})} \\ &\le C(n, \lambda, \Lambda)\{1 + \|f\|_{L^n(B_1)}\}. \end{aligned}$$

Since $u - h = 0$ on $\partial B_{\frac{3}{4}}$, if we let x_0 be the intersection point of the line connecting 0 and x with $\partial B_{\frac{3}{4}}$. We have

$$\begin{aligned} |u(x) - h(x)| &\le |u(x) - u(x_0)| + |h(x) - h(x_0)| \\ &\le C\delta^{\frac{\alpha}{2}}(1 + \delta^{\frac{\alpha}{2}})\{1 + \|f\|_{L^n(B_1)}\} \\ &\le C\delta^{\frac{\alpha}{2}}\{1 + \|f\|_{L^n(B_1)}\} \end{aligned}$$

for $\forall x \in \partial B_{\frac{3}{4}-\delta}$. We claim for any $0 < \delta < 1$,

$$|D^2h|_{L^\infty(B_{\frac{3}{4}-\delta})} \le C\delta^{\frac{\alpha}{2}-2}\{1 + \|f\|_{L^n(B_1)}\}.$$

In fact, for any $x_0 \in B_{\frac{3}{4}-\delta}$, we apply interior C^2-estimate to $h - h(x_1)$ in $B_\delta(x_0) \subset B_{\frac{3}{4}}$ for some $x_1 \in \partial B_\delta(x_0)$ and obtain

$$\begin{aligned} |D^2h(x_0)| &\le C\delta^{-2} \sup_{B_\delta(x_0)} |h - h(x_1)| \\ &\le C\delta^{-2}\delta^{\frac{\alpha}{2}}\{1 + \|f\|_{L^n(B_1)}\}. \end{aligned}$$

Note that $u - h$ is a viscosity solution of

$$a_{ij}D_{ij}(u - h) = f - (a_{ij} - a_{ij}(0))D_{ij}h \equiv F \qquad in B_{\frac{3}{4}}.$$

Then by Alexandroff maximum principle(Theorem 5.8 in H-L) and previous estimates, we have

$$
\begin{aligned}
|u-h|_{L^\infty(B_{\frac{3}{4}-\delta})} &\leq |u-h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} + C\|F\|_{L^n(B_{\frac{3}{4}-\delta})} \quad (\star)\\
&\leq |u-h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} + C|D^2h|_{L^\infty(B_{\frac{3}{4}-\delta})}\|a_{ij}-a_{ij}(0)\|_{L^n(B_{\frac{3}{4}})} + C\|f\|_{L^n(B_1)}\\
&\leq C(\delta+\delta^{\frac{\alpha}{2}-2}\varepsilon)\{1+\|f\|_{L^n(B_1)}\} + C\|f\|_{L^n(B_1)}.
\end{aligned}
$$

Take $\delta=\varepsilon^{\frac{1}{2}}<\frac{1}{4}$ and $\gamma=\frac{\alpha}{4}$, this finishes the proof.

Remark 3.2. *In the proof of above lemma, we use the interior C^2-estimate for second order elliptic equations with constant coefficients. In fact, it is a generalization of Proposition 1.13 in H-L. We give a short proof in the following.*

Theorem 3.3. *Suppose that $h\in C(\overline{B}_R)\cap C^\infty(B_R)$ satisfies $a_{ij}D_{ij}h=0$ in B_R, where $A=(a_{ij})_{n\times n}$ is a constant, positive definite matrix with minimal eigenvalue λ and maximal eigenvalue Λ. Then we have*

$$|D^2h(0)|\leq \frac{C(n,\lambda,\Lambda)}{R^2}\max_{\overline{B}_R}|h|$$

Proof : Let $y=Bx$, where B is an invertible matrix to be determined. Define $\overline{h}(y)=h(x)$, simple calculation yields: $(D_x^2h)=B^T(D_y^2\overline{h})B$. Hence we have

$$tr(AD_x^2h)=tr(AB^TD_y^2\overline{h}B)=tr(BAB^TD_y^2\overline{h})=0.$$

A is positive definite with eigenvalues $0<\lambda=\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n=\Lambda$. Then there exists an orthogonal matrix P such that

$$PAP^T=diag(\lambda_1,\cdots\cdots,\lambda_n).$$

If we let $B=diag(\frac{1}{\sqrt{\lambda_1}},\cdots\cdots,\frac{1}{\sqrt{\lambda_n}})P$, then $BAB^T=I$ and $\Delta_y\overline{h}=0$. Note that after the linear transformation B, B_R becomes an ellipsoid which has the equation

$$\sum_{i=1}^n\lambda_iz_i^2<R^2.$$

It is easy to find that the ellipsoid has a inscribed ball: $\sum_{i=1}^n z_i^2<\frac{R^2}{\Lambda}$. Hence by Proposition 1.13 in H-L, we deduce that

$$|D_y^2\overline{h}(0)|\leq\frac{C(n)\Lambda}{R^2}\max_{\overline{B}_{R/\sqrt{\Lambda}}}|h|.$$

Change y back to x and notice that the pre-image of $\overline{B}_{R/\sqrt{\Lambda}}$ is in $\overline{B}_R$, this gives the consequence immediately.

Remark 3.4. *In the proof of this lemma, we apply the Alexandroff maximum principle, but in fact it can't be used directly, we should define two auxiliary functions as follows:*

$$g_1(x) = |u-h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} - (u-h),$$
$$g_2(x) = |u-h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} + (u-h).$$

Then apply the Alexandroff maximum principle to g_1, g_2 in $B_{\frac{3}{4}-\delta}$ respectively and after a simple argument, we can get $(\star)$.

4. Schauder Estimates

Definition 4.1. *A function g is Hölder continuous at 0 with exponent α in L^n-sense if*

$$[g]_{C^\alpha_{L^n}(0)} \equiv \sup_{0<r<1} \frac{1}{r^\alpha}\Big(\frac{1}{|B_r|}\int_{B_r} |g-g(0)|^n\Big)^{\frac{1}{n}} < \infty.$$

Theorem 4.1. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f$$

in B_1, assume $\{a_{ij}\}$ is Hölder continuous at 0 with exponent α in L^n-sense for some $\alpha \in (0,1)$. If f is Hölder continuous at 0 with exponent α in L^n-sense, then u is $C^{2,\alpha}$ at 0. Moreover there exists a polynomial P of degree 2 such that

$$|u-P|_{L^\infty(B_r(0))} \le C_* r^{2+\alpha},$$
$$|P(0)| + |DP(0)| + |D^2P(0)| \le C_*,$$
$$C_* \le C(|u|_{L^\infty(B_1)} + |f(0)| + [f]_{C^\alpha_{L^n}(0)})$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \alpha$ and $[a_{ij}]_{C^\alpha_{L^n}(0)}$.

Proof : First we assume that $f(0) = 0$. For that we may consider $v = u - b_{ij}x_ix_jf(0)/2$ for some constant matrix (b_{ij}) such that $a_{ij}(0)b_{ij} = 1$. By scaling we also assume that $[a_{ij}]_{C^\alpha_{L^n}(0)}$ is small. In fact, if we define

$$\tilde{u}(y) = u(Ry),$$
$$\widetilde{a}_{ij}(y) = a_{ij}(Ry),$$
$$\tilde{f}(y) = R^2 f(Ry).$$

Then $\tilde{u} \in C(B_R)$ is a viscosity solution of $\widetilde{a}_{ij}D_{ij}\tilde{u} = \tilde{f}$. Hence

$$[\widetilde{a}_{ij}]_{C^\alpha_{L^n}(0)} = \sup_{0<r<1} \frac{1}{r^\alpha}\Big(\frac{1}{|B_r|}\int_{B_r} |\widetilde{a}_{ij}(y) - \widetilde{a}_{ij}(0)|^n\Big)^{\frac{1}{n}}$$
$$= R^\alpha \sup_{0<r<1} \frac{1}{(Rr)^\alpha}\Big(\frac{1}{|B_{Rr}|}\int_{B_{Rr}} |a_{ij}(z) - a_{ij}(0)|^n\Big)^{\frac{1}{n}}.$$

Then R small implies $[\widetilde{a}_{ij}]_{C^{\alpha}_{L^n}(0)}$ is small. Next by considering

$$\frac{u}{|u|_{L^\infty(B_1)} + \frac{1}{\delta}[f]_{C^{\alpha}_{L^n}(0)}}$$

for $\delta > 0$, we may assume $|u|_{L^\infty(B_1)} \leq 1$ and $[f]_{C^{\alpha}_{L^n}(0)} \leq \delta$.
In the following we prove that there is a constant $\delta > 0$, depending only on n, λ, Λ and α, such that if $u \in C(B_1)$ is a viscosity solution of $a_{ij}D_{ij}u = f$ in B_1, with

$$|u|_{L^\infty(B_1)} \leq 1, \quad [a_{ij}]_{C^{\alpha}_{L^n}(0)} \leq \delta, \quad \left(\frac{1}{|B_r|}\int_{B_r} |f|^n\right)^{\frac{1}{n}} \leq \delta r^\alpha$$

for any $0 < r < 1$, then there exists a polynomial P of degree 2 such that for any $0 < r < 1$

$$|u - P|_{L^\infty(B_r(0))} \leq C r^{2+\alpha} \tag{4.1}$$

and

$$|P(0)| + |DP(0)| + |D^2P(0)| \leq C \tag{4.2}$$

for some positive constant C depending only n, λ, Λ and α.
We claim that there exists $0 < \mu < 1$, depending only on n, λ, Λ and α, and a sequence of polynomial of degree 2

$$P_k(x) = a_k + b_k \cdot x + \frac{1}{2}x^t C_k x$$

such that for any $k = 0, 1, 2, \cdots$

$$a_{ij}(0)D_{ij}P_k = 0, \tag{4.3}$$

$$|u - P_k|_{L^\infty(B_{\mu^k})} \leq \mu^{k(2+\alpha)} \tag{4.4}$$

and

$$|a_k - a_{k-1}| + \mu^{k-1}|b_k - b_{k-1}| + \mu^{2(k-1)}|C_k - C_{k-1}| \leq C\mu^{(k-1)(2+\alpha)}, \tag{4.5}$$

where $P_0 = P_{-1} \equiv 0$ and C is a positive constant, depending only on n, λ, Λ and α.
We first prove that Theorem 4.1 follows from (4.3), (4.4) and (4.5). It is easy to see that a_k, b_k and C_k converge and the limiting polynomial

$$P(x) = a_\infty + b_\infty \cdot x + \frac{1}{2}x^t C_\infty x$$

satisfies

$$\begin{aligned}
|P_k(x) - P(x)| &\leq \sum_{m=k}^{\infty} |P_{m+1}(x) - P_m(x)| \\
&\leq C \sum_{m=k}^{\infty} (\mu^{(\alpha+2)k} + |x|\mu^{(\alpha+1)k} + |x|^2\mu^{\alpha k}) \\
&\leq C\mu^{(2+\alpha)k}
\end{aligned}$$

for any $|x| \le \mu^k$. Hence we have for $|x| \le \mu^k$,

$$|u(x) - P(x)| \le |u(x) - P_k(x)| + |P_k(x) - P(x)| \le C\mu^{(2+\alpha)k}.$$

Hence for any $x \in B_1$, $\exists k_0 \ge 0$ such that $\mu^{k_0+1} \le |x| \le \mu^{k_0}$, then

$$|u(x) - P(x)| \le C\mu^{k_0(2+\alpha)} = \frac{C}{\mu^{2+\alpha}}\mu^{(k_0+1)(2+\alpha)} \le C|x|^{2+\alpha},$$

which means (4.1).
To prove (4.2), observe that

$$|a_\infty| \le \sum_{k=1}^{\infty} |a_k - a_{k-1}| \le C\sum_{k=1}^{\infty} \mu^{(k+1)(2+\alpha)} = \frac{C}{1-\mu^{2+\alpha}} = C(n, \lambda, \Lambda, \alpha).$$

$|b_\infty|, |C_\infty|$ can be calculated similarily, hence (4.2) is get under the claim.
Now we prove $(4.3), (4.4)$ and (4.5). Clearly $(4.3), (4.4)$ and (4.5) hold for $k = 0$. Assume they hold for $k = 0, 1, 2, \dots, l$, we prove for $k = l + 1$. Consider the function

$$\tilde{u}(y) = \frac{1}{\mu^{l(2+\alpha)}}(u - P_l)(\mu^l y)$$

for $y \in B_1$. Then $\tilde{u} \in C(B_1)$ is a viscosity solution of $\widetilde{a}_{ij} D_{ij}\tilde{u} = \tilde{f}$ in B_1 with

$$\begin{aligned} \widetilde{a}_{ij}(y) &= a_{ij}(\mu^l y), \\ \tilde{f}(y) &= \frac{1}{\mu^{l\alpha}}\{f(\mu^l y) - a_{ij}(\mu^l y)D_{ij}P_l\}. \end{aligned}$$

Now we check that $\tilde{u}$ satisfies the assumptions of Lemma 3.1. For that we calculate

$$\begin{aligned} \|\widetilde{a}_{ij}(y) - \widetilde{a}_{ij}(0)\|_{L^n(B_1)} &\le [a_{ij}]_{C^\alpha_{L^n}(0)} \cdot w_n^{\frac{1}{n}} \cdot \mu^{l\alpha} \\ &\le w_n^{\frac{1}{n}}\delta, \end{aligned}$$

$$|\tilde{u}|_{L^\infty(B_1)} \le 1,$$

$$\begin{aligned} \|\tilde{f}\|_{L^n(B_1)} &\le \frac{1}{\mu^{l\alpha}}\|f(\mu^l y)\|_{L^n(B_1)} + \sup_{B_{\mu^l}} |D^2 P_l| \|a_{ij}(\mu^l y) - a_{ij}(0)\|_{L^n(B_1)} \\ &\le w_n^{\frac{1}{n}} \cdot \delta + \frac{1}{1-\mu^\alpha} \cdot w_n^{\frac{1}{n}} \cdot \delta \\ &\le \frac{2w_n^{\frac{1}{n}}\delta}{1-\mu^\alpha}. \end{aligned}$$

We take $\varepsilon = \frac{2w_n^{\frac{1}{n}}\delta}{1-\mu^\alpha} < \frac{1}{16}$ in Lemma 3.1, there exisits a function $h \in C(\overline{B}_{\frac{3}{4}})$ with $\tilde{a}_{ij}(0)D_{ij}h = 0$ in $B_{\frac{3}{4}}$ and $|h| \le 1$ in $B_{\frac{3}{4}}$, such that

$$|\tilde{u} - h|_{L^\infty(B_{\frac{1}{2}})} \le C_1\{\varepsilon^\gamma + \varepsilon\} \le 2C_1\varepsilon^\gamma,$$

where $C_1 = C_1(n, \lambda, \Lambda)$ and $\gamma = \gamma(n, \lambda, \Lambda)$. Write

$$\tilde{P}(y) = h(0) + Dh(0) + \frac{y^t D^2 h(0) y}{2}.$$

Then by interior estimate similarily as Proposition 1.31 in H-L, we have

$$\begin{aligned}
|\tilde{u} - \tilde{P}|_{L^\infty(B_\mu)} &\leq |\tilde{u} - h|_{L^\infty(B_\mu)} + |h - \tilde{P}|_{L^\infty(B_\mu)} \\
&\leq 2C_1 \varepsilon^\gamma + C_2 \mu^3 \\
&= 2C_1 \left(\frac{2w_n^{\frac{1}{n}} \delta}{1-\mu^\alpha}\right)^\gamma + C_2 \mu^3 \\
&\leq \mu^{2+\alpha},
\end{aligned}$$

where $C_2 = C_2(n, \lambda, \Lambda)$ and by choosing μ small and then ε small accordingly. Rescaling back we have

$$|u(x) - P_l(x) - \mu^{l(2+\alpha)} \tilde{P}(\mu^{-l} x)| \leq \mu^{(l+1)(2+\alpha)}$$

for any $x \in B_{\mu^{l+1}}$. If we define

$$P_{l+1}(x) = P_l(x) + \mu^{l(2+\alpha)} \tilde{P}(\mu^{-l} x),$$

next we prove that $P_{l+1}(x)$ satisfies (4.3), (4.4) and (4.5).
First, (4.3) and (4.4) is satisfied obviously. To prove (4.5), notice that

$$\begin{aligned}
|a_{l+1} - a_l| + \mu^l |b_{l+1} - b_l| + \mu^{2l} |C_{l+1} - C_l| &\leq \mu^{l(2+\alpha)} (|h(0)| + |Dh(0)| + |D^2 h(0)|) \\
&\leq C(n, \lambda, \Lambda, \alpha) \mu^{l(2+\alpha)}.
\end{aligned}$$

The last inequality follows from Theorem 3.3.

Remark 4.2. *Review the proof above, keep in mind that what we really need is the existence of δ and μ. Both of them must be independent of l in the iteration process, we see that δ and μ must satisfies the following inequalities:*

$$\frac{2w_n^{\frac{1}{n}} \delta}{1-\mu^\alpha} < \frac{1}{16}, \tag{4.6}$$

$$\mu \leq \frac{1}{2}, \tag{4.7}$$

$$2C_1 \left(\frac{2w_n^{\frac{1}{n}} \delta}{1-\mu^\alpha}\right)^\gamma + C_2 \mu^3 \leq \mu^{2+\alpha}. \tag{4.8}$$

if we let

$$2C_1 \left(\frac{2w_n^{\frac{1}{n}} \delta}{1-\mu^\alpha}\right)^\gamma + C_2 \mu^3 = \frac{1}{2} \mu^{2+\alpha},$$

we can deduce

$$\delta = \frac{1}{2w_n^{\frac{1}{n}}}(\frac{\mu^{2+\alpha}}{4C_1})^{\frac{1}{\gamma}}(1-\mu^\alpha),$$
$$\mu < \min\{\frac{1}{2}, (\frac{1}{2C_2})^{\frac{1}{1-\alpha}}, (\frac{4C_1}{16^\gamma})^{\frac{1}{2+\alpha}}\}.$$

Remark 4.3. *The generalization of proposition 1.31 in H-L. The proof is similarily as the argument in Theorem 3.3, by choosing the cutoff function η appropriately.*

Remark 4.4. *Think about where the conditions apply in the proof of Theorem 4.1 and which assumption fails if we want to give $C^{1,\alpha}$ estimates. How to modify the condition in order to make the similar proof work?*

Theorem 4.5. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f$$

in B_1. Then for any $\alpha \in (0,1)$, there exists an $\theta > 0$ depending only on n, λ, Λ and α, such that if

$$\sup_{0<r\le 1}(\frac{1}{|B_r|}\int_{B_r}|a_{ij}-a_{ij}(0)|^n)^{\frac{1}{n}} \le \theta,$$
$$\sup_{0<r<1} r^{1-\alpha}(\frac{1}{|B_r|}\int_{B_r}|f|^n)^{\frac{1}{n}} < \infty,$$

then u is $C^{1,\alpha}$ at 0; That is, there exists an affine function L such that

$$|u-L|_{L^\infty(B_r(0))} \le C_* r^{1+\alpha},$$
$$|L(0)| + |DL(0)| \le C_*$$

and

$$C_* \le C(|u|_{L^\infty(B_1)} + \sup_{0<r<1} r^{1-\alpha}(\frac{1}{|B_r|}\int_{B_r}|f|^n)^{\frac{1}{n}}),$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \alpha$.

Remark 4.6. *According to Theorem 4.1 and Remark 4.4, the proof is actually similar to that of Theorem 4.1, we don't give a concrete proof here, but it will be discussed in the seminar if necessary.*